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ABSTRACT

Within this single module there are two approaches to this brief survey of logic. Since most geometry textbooks fail to give an adequate discussion of logic, a "textbook" treatment of the subject has been included. This is found as explanations interspersed in the exercises and these can be used as a textbook approach. However, also included is an activities approach which appears at the beginning of each section. (Author/MK)

GEOMETRY MODULE FOR USE
IN A
MATHEMATICS LABORATORY SETTING

LOGIC

by
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A Publication of

University of Denver
Mathematics Laboratory
Regional Center for
Pre-College Mathematics

Dr. Ruth I. Hoffman, Director

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Objectives

1. The student will be able to determine the truth value of a given statement.
2. The student will be able to determine the negation of a statement and recognize the equivalence of two statements.
3. The student will be able to determine whether a given argument is valid.
4. The student will be able to form simple proofs.

Overview

Within this single module there are two approaches to this brief survey of logic. Since most geometry textbooks fail to give an adequate discussion of logic we have included a "textbook" treatment of the subject. This is found as explanations interspersed in the exercises and these can be used as a textbook approach. However, also included is an activities approach which appears at the beginning of each section. We believe this allows you the options necessary to vary the presentation of the material, thus maintaining student interest. For example, a class of strong students may tire of the activities and wish to get on with the basic material. In this case the text material alone may do a job better. A slower group might shy away from a textbook approach and concentrate only on the activities.

Contents

Activities

Exercises

1. Statements
2. Equivalence and Negation
3. Conjunction and Disjunction
4. Compound Statements and DeMorgan's Laws
5. Implications
6. Converse, Biconditional, Inverse, Contrapositive
7. Tautologies and Contradictions.
8. Proofs

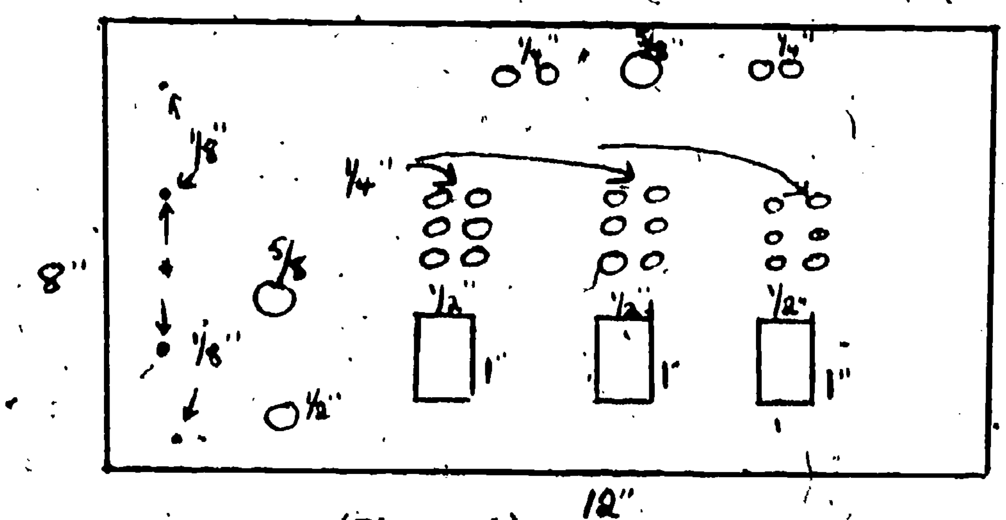
Of these Activities 2,3,4,6 and 7 make use of the logic circuit boards potential to model logic expressions. Activity 1 is an individual work sheet, 5 is a whole class activity and 8 is for small groups.

Materials

1. Logic circuit boards (At least one board for four students). These may be purchased or built. Plans for construction of a logic board are included here.

Logic Board Construction

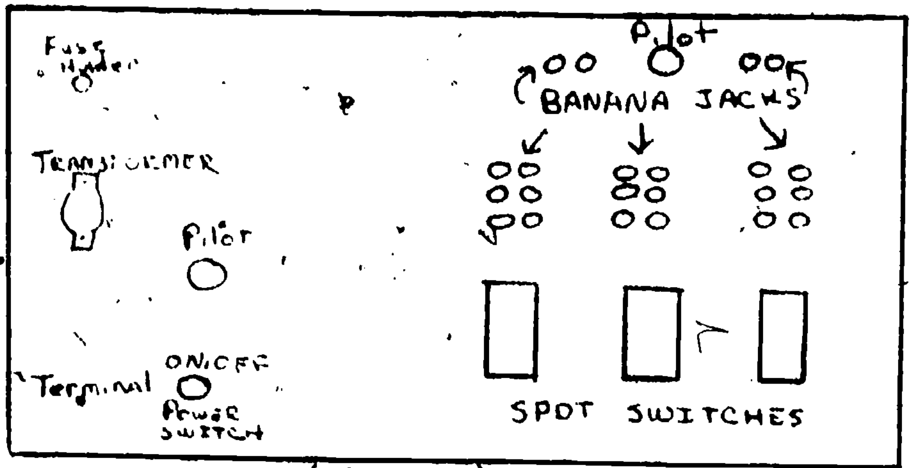
I. Drill holes as indicated by figure I. The rectangular holes for the SPDT slide switches may vary with the size of the switches you purchased.



(Figure 1)

*This distance must correspond to the distance between mounting holes of the 117v/60cy - 6.3V.1.2A transformer.(about 2½").

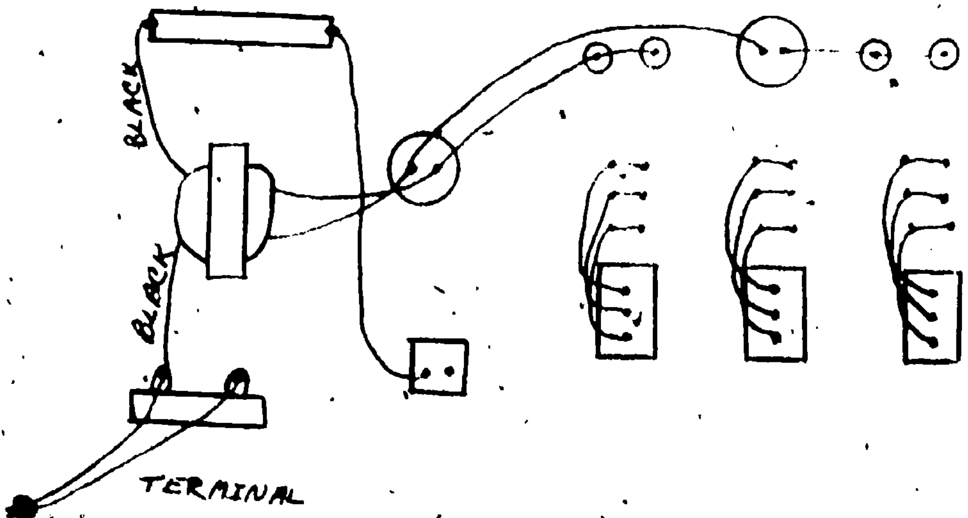
II Mount the components on the 8"x12" board as indicated by figure 2.



(Figure 2)

The transformer, fuse holder and terminal strip are mounted on the under side of the board.

III. Study Figure 3 before completing the wiring of your L. C. B.



(Figure 3)

IV. You may design your own case for your L. C. B. The following list of materials may be of help to your when constructing the case.

Materials - L. C. B. case

2 - 8 1/2" x 15" x 1/8 (top bottom)

2 - 5" x 8" x 1/4"

2 - 5" x 15" x 1/4"

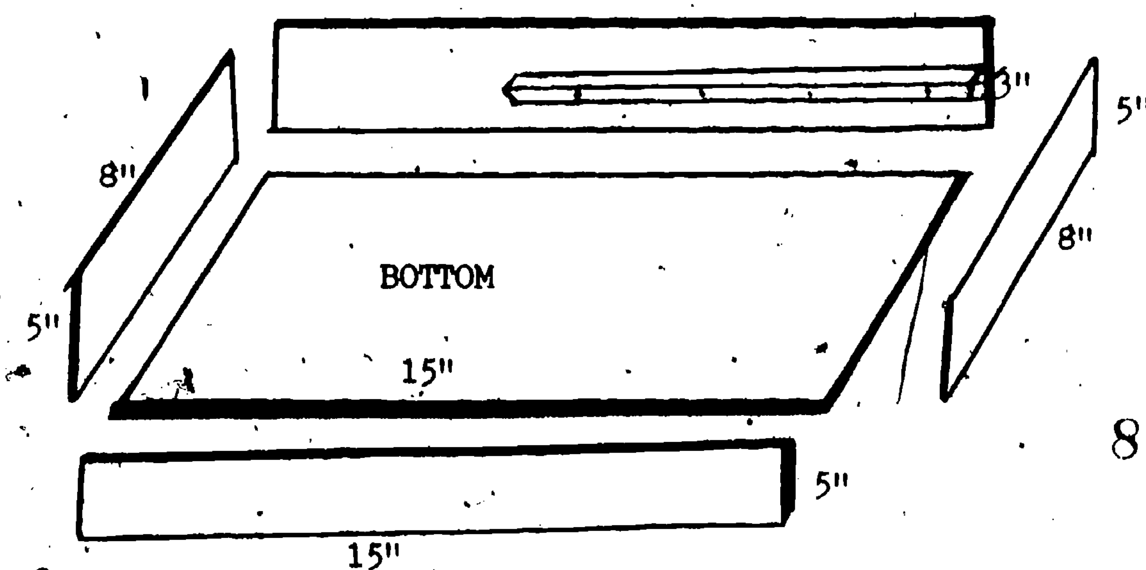
1 - 8" x 4" x 1/8"

1 - 8" x 12" x 1/8"

2 - 1/2" x 1/2" x 12"

Assorted bolts, nuts, nails, screws, glue.

(Figure 4)



V. After completing the box, use a table saw to saw the lid off 1" from the top. Attach lid with appropriate hinges, secure with latoh.

L.C.B. Materials

1 - 117V/60cp - 6.3v.1.2A Transformer

3.- Spot Slide Switches

2.- Pilot Lights

1 - Fuse holder

1 - fuse (1.A)

1 - Terminal (3 terminals)

22 - Banana Jacks

1 - SPST Toggle Switch

1 - 6' Extension Cord

If after reading this and you would like to purchase rather than build your L.C.B., please contact BBR Enterprises, Littleton High School, c/o Jim Reed.

Building Blocks of Logic - Statements

Teaching Suggestions

1. The activity is a true-false test designed to introduce the student to logic. A class discussion of the test - interpretation will give the teacher opportunity to preview the module. Example: No. 14 leads to the question: "What kind of sentences can be on a true-false test? The students will be ready to define "statement" in the exercises. Example: No. 15 How shall we interpret the negation of a statement?

2. As the ideas touched on here are clarified in the module the teacher may want to come back to this activity.

Materials

1. Activity card for each student.

   Work done by each student:

   a. Activity card

   b. Complete exercises

Exercise Answers L-3

1. Yes
2. No
3. Yes
4. Yes
5. Yes
6. Yes
7. Yes
8. No
9. Yes
10. Yes
11. No
12. Yes

11

"The Circuit Board"

Teaching Suggestions

1. There should be a circuit board available for each group of two to four students.
2. The activity is simple and requires only a small amount of time to complete. It's purpose is to acquaint the student with the circuit board.
3. Since the real thrust of the material on equivalent statements and negations is contained in the exercises, it might be advantageous to do the exercises before the activity.

Materials

1. Circuit board for each group.
2. Activity card for each student.

Work done by each student:

1. Participate in circuitboard activity
2. Complete exercises.

1. yes
2. no
3. yes
4. no
5. no
6. yes
7. Solution set
8.

p	q	r
T	T	T
F	F	F

a) $p \leftrightarrow r$

b) transitive property

9. no
10. yes
11. no
12. yes
13. no
14. no

1.

P	P	P
T	F	T
F	T	F

a) equivalent

2. a) y > 5

b) $x + 4 \neq 9$ or $x \neq 5$

c) $2y + 4 \neq 8$ or $x \neq 2$

d) x = 9

e) y 5

f) 3y y + 5 or y 5/2

3.

p	q	r	s
T	F	T	F
F	T	F	T

a) s is a negation of p

4.

p	q	r	s
T	T	F	F
F	F	T	T

a) p eqivalent to q

b) r equivalent to s

c) equivalent

5. $x \geq 2$

6. It is false that all of us are hungry.

Not all of us are hungry

One of us is not hungry

7. It is false that some people sleep in class.

All people sleep in class.

No one sleeps in class.

8. It is false that no student likes Geometry.

One student likes Geometry.

Some students like Geometry.

9. It is false that Richard is never serious

Sometimes Richard is serious.

Richard is always serious.

10. It is false that none of us will be here tomorrow.

Some of us will be here tomorrow

One of us will be here tomorrow.

In questions 6 – 10 there exist other possibilities.

Teaching Suggestions

1. The activity again involves small groups working on circuit boards.
2. It's important that each member of the group understand the "and" and "or" circuits on order to do activity Cards 4,6,7.
3. The exercises contain a development of conjunction and disjunction. The teacher may wish to use this material as a class discussion before assigning the exercises, or assign the exercises first (individually or to small groups) and then follow up with a review discussion after the exercises are completed. Stronger groups will need little or no teahcer involvement.

Materials

1. Circuit board for each group.
2. Activity card for each student.

Work done by each student:

1. Participate in circuitboard activity. Each student learns to wire And and Or Circuit.
2. Complete exercises.

Conjunction and Disjunction

And and Or statements

1.

p	q	$p \wedge q$
T	T	T
F	F	F
F	T	F
F	F	F

2. Both statements are true

3.

p	q	p q
T	T	T
T	F	T
F	T	T
F	F	F

4. Both statements are false.

5. a) Mastedons were big and men don't walk up walls.

   b) Mastedons were big or men don't walk up walls.

6. a) Every triangle has three sides and every square has four right angles.

   b) Every triangle has three sides or every square has four right angles.

   c) T

   d) T

7. a) A pentagon has five sides and a square has three diagonals.

   b) A pentagon has five sides or a square has three diagonals.

   c) F

   d) T

8. b

9. c

10. a) F

    b) F

11. a) T

    b) T

    c) F

    d) F

    e) F

    f) F

More Complex Circuits - Compound Statements and DeMorgan's Laws

Teaching Suggestions

1. The activity involves small groups working with circuit boards.
2. Suggest the students draw the circuit before attempting to wire it.
3. The activity may involve more than one class period to complete. As a result, the teacher may want to assign only part of the activity.
4. The exercises develop a truth table step by step. The teacher may wish to do this for the class.

Materials

1. Circuit board for each group.
2. Activity card for each student.

Work done by student:

1. Participate in circuit board activity.
2. Complete exercises.

Exercise Answers L18 - L20

Compound Statements

1.

p	q	$\sim p$	$\sim q$	$(\sim p \vee \sim q)$	$\sim(\sim p \vee \sim q)$
T	T	F	F	F	T
T	F	F	T	T	F
F	T	T	F	T	F
F	F	T	T	T	F

2.

p	q	$\sim q$	$p \vee \sim q$
T	T	F	T
T	F	T	T
F	T	F	F
F	F	T	T

3.

p	q	$\sim q$	$p \wedge \sim q$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

4.

p	q	$\sim p$	$\sim p \vee q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

5.

p	q	$\sim p$	$\sim q$	$\sim p \vee \sim q$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

6.

p	q	$\sim p$	$\sim q$	$(p \vee \sim q)$	$(q \vee \sim p)$	$(p \vee \sim q) \wedge (q \vee \sim p)$
T	T	F	F	T	T	T
T	F	F	T	T	F	F
F	T	T	F	F	T	F
F	F	T	T	T	T	T

7.

p	q	$(p \vee q)$	$\sim(p \vee q)$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

8.

p	q	$\sim p$	$\sim q$	$\sim p \wedge \sim q$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

9.

p	q	$(p \wedge q)$	$\sim(p \wedge q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

10.

p	q	$\sim p$	$\sim q$	$\sim p \vee \sim q$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

11. Bill will not work or Jim will not work.

12. It is false that Joe graduates in June and Jerry will be a Junior.

Joe will not graduate in June or Jerry will not be a Junior.

13.

p	q	r	$(p \wedge q)$	$(p \wedge q) \vee r$
T	T	T	T	T
T	T	F	T	T
T	F	T	F	T
T	F	F	F	F
F	T	T	F	T
F	T	F	F	F
F	F	T	F	T
F	F	F	F	F

Who broke what promise? Implications

Teaching Suggestions

1. The activity consists of a play in three acts. The first two acts show the truth value of an implication, the third prepares students for defining an implication as an Or statement.
2. Select a group of students to present each act. Encourage them to embellish parts as much as they like, as long as they don't change the basic situation.
3. The success of this activity depends on the teacher's enthusiasm. It's important that the class be led to the right response in each situation.

Work done by the student:

1. Participate in drama
2. Complete exercises

Materials

1. Activity Cards
2. Board or overhead for recording class responses.

1.

p	q	$\sim p$	$\sim p \vee q$
T	T	F	T
T	F	F	F
F	T	T	T

2.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

3. If it rains, then we'll go home early.

4. If we win there will be no school tomorrow.

5. $4x=20$ implies $x=5$

6. We're going only if the weather is nice.

7. To be eligible it's necessary to keep your grades up.

8. If I eat less, then I lose weight.

9. If the cat's away, then the mouse will play.

10. If I finish my home work in time, then I'll be there.

11. If you like a course then you do well in it.

12. If you are a teacher then you must attend the meeting.

13. If you are a student then you can join the club.

14. If you are a cheerleader, then you are a junior.

15. If you live in Denver, then you live in Colorado.

16. If a polygon is a triangle, then it has three sides.

17. If you live in Denver, then you live in Colorado.

18. If you live in Denver, then you live in Colorado.

Variations of a Promise
Converse, Biconditional, Inverse, Contrapositive

Teaching Suggestions

1. Small groups working on circuit board. This activity will reinforce the definition of implication and introduce the definition of converse, Inverse, and contrapositive. However, as usual in this module, the activity can be omitted without loss of material content.

Materials

1. Circuit board for each group.
2. Activity card for each student.

Work done by student:

1. Participate in Circuit board activity.
2. Complete Exercises.

1. If I lose weight, then I eat less.
2. If the mouse will play, then the cat's away.
3. If I'm there, then I finished my homework in time.
4. If you do well in it, then you like a course.
5.

p	q	p → q	q → p
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

6. No (not when p is false and q is true).
7. Answers will vary. (give an example).
8. Answers will vary.
9.

p	q	p → q	q → p	(p → q) ∧ (q → p)
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

10. I can wake up is equivalent to I can go to sleep.
11. I can wake up is necessary and sufficient for my being able to go to sleep.
12. A polygon has three sides if and only if a polygon is a triangle.
13. A polygon has three sides is necessary and sufficient for a polygon being a triangle.
14. I live in Denver is equivalent to I live in the Mile High City.
15. I live in Denver if and only if I live in the Mile High City.

1. If you're not a teacher, then you must not attend the meeting.
2. If you're not a student, then you can't join the club.
3. If you're not a cheerleader, then you're not a Junior.
4. If you don't live in Denver, then you don't live in Colorado.
5.

p	q	$p \rightarrow q$	$\sim p$	$\sim q$	$\sim p \rightarrow \sim q$
T	T	T	F	F	T
T	F	F	F	T	T
F	T	T	T	F	F
F	F	T	T	T	T

6. No (not when p is false and q is true).
7. Answers will vary. (give an example)
8. Answers will vary.

1. If a polygon does not have three sides then it is not a triangle.
2. If you don't live in Colorado, then you don't live in Denver.
3.

p	q	$p \to q$	$\sim p$	$\sim q$	$\sim q \to \sim p$
T	T	T	F	F	T
T	F	F	F	T	F
F	T	T	T	F	T
F	F	T	T	T	T

4. yes
5. equivalent.

Teaching Suggestions

1. The activity again involves small groups working on circuit boards.
2. Students may be tired of truth tables by now and dividing the exercices among small groups would lessen the work.

Materials

1. Circuit board for each group.
2. Activity card for each student.

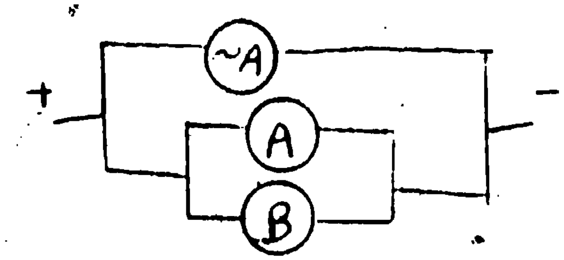
Work done by each student:

1. Participate in circuit board activity
2. Complete exercis

Exercises Answers for L-34

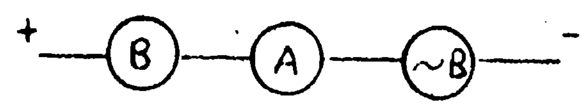
Draw a circuit for $A \rightarrow A \vee B$

The circuit is:



Draw a circuit for $B \wedge (A \wedge \sim B)$.

The circuit is:



1.

p	q	$(p \vee q)$	$p \rightarrow (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

2.

p	q	$\sim q$	$(p \wedge \sim q)$	$q \wedge (p \wedge \sim q)$
T	T	F	F	F
T	F	T	T	F
F	T	F	F	F
F	F	T	F	F

3. Tautology

p	$\sim p$	$p \vee \sim p$
T	F	T
F	T	T

4. Contradiction

p	$\sim p$	$p \wedge \sim p$
T	F	F
F	T	F

5. Tautology

p	q	$(p \wedge q)$	$(p \wedge q) \rightarrow p$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

6. Neither

p	q	$(p \wedge q)$	$p \rightarrow (p \wedge$
T	T	T	T
T	F	F	F
F	T	F	T
F	F	F	T

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Tautology

r	s	$(r \vee s)$	$\sim(r \vee s)$	$\sim r$	$\sim s$	$\sim r \wedge \sim s$	$\sim(r \vee s) \leftrightarrow \sim r \wedge \sim s$
T	T	T	F	F	F	F	T
T	F	T	F	F	T	F	T
F	T	T	F	T	F	F	
F	F	F	T	T	T	T	

8. Tautology

p	q	$(p \leftrightarrow q)$	$p \wedge (p \rightarrow q)$	$[p \wedge (p \rightarrow q)] \rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

9. Neither

p	q	$\sim q$	$(p \wedge \sim q)$	$\sim(p \wedge \sim q)$	$(p \wedge q)$	$\sim(p \wedge \sim q) \vee (p \wedge q)$
T	T	F	F	T	T	T
T	F	T	T	F	F	F
F	T	F	F	T	F	T
F	F	T	F	T	F	T

10. Contradiction

p	q	$\sim p$	$\sim q$	$(p \vee \sim p)$	$(q \wedge \sim q)$	$(p \vee \sim p) \rightarrow (q \wedge \sim q)$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	T	F	F

11. Tautology

p	q	$(p \to q)$	$\sim p$	$\sim q$	$(\sim q \to \sim p)$	$(p \to q) \leftrightarrow (\sim q \to \sim p)$
T	T	T	F	F	T	T
T	F	F	F	T	F	T
F	T	T	T	F	T	T
F	F	T	T	T	T	T

12. Neither.

p	q	$\sim q$	$(p \wedge \sim q)$	$(p \wedge q)$	$(p \wedge \sim q) \vee (p \wedge q)$
T	T	F	F	T	T
T	F	T	T	F	T
F	T	F	F	F	F
F	F	T	F	F	F

13. Tautology

p	q	r	$(p \to q)$	$(q \to r)$	$(p \to r)$	$(p \to q) \wedge (q \to r)$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	F	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	T	F	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

1. Divide the class into small groups to work on this activity. Instruct them to agree on the validity of a conclusion before they write down their answer.

2. Correct results for the activity.

   I. Conclusion not valid.

      Babies cannot manage crocodiles.

   II. Conclusion valid.

   III. Conclusion valid .

   IV. Conclusion not valid.

      No marmots take in the Post.

   V. Conclusion not valid.

      No gray ducks in this village wear lace collars

   VI. Conclusion valid.

   VII. Conclusion not valid.

      An egg of the Great Auk is not to be had for a song.

   VIII. Conclusion not valid.

      No bird in this aviary lives on mince pies.

   IX. Conclusion is valid.

   X. Conclusion: Rainy days are always cloudy.

3. This activity is included in order to introduce the need for determining the validity of an argument. Therefore disagreement about the conclusions drawn in various arguments are to be encouraged. It is hoped that the methods introduced in the exercises can be used to resolve this type of disagreement.

1. $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$ $[p \wedge (p \rightarrow q)] \rightarrow q$

2. Hypothesis

p: I am a freshman student.

and $p \rightarrow q$: If a student is a freshman, then he may attend the freshman class party.

Conclusion:

q: I can attend the Freshman class party

Tautology

$[p \wedge (p \rightarrow q)] \rightarrow q$

3. Hypothesis

$p \rightarrow q$: If I eat pickles with ice cream, I dream about lions and tigers.

and $q \rightarrow r$: If I dream about lions and tigers, I wake up scared.

Conclusion:

$p \rightarrow r$: If I eat pickles with ice cream, I wake up scared.

Tautology

$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$

4. Hypothesis

p: This is the fourth week of school.

and $p \rightarrow q$: If this is the fourth week of school, then this is the week my theme is due.

Conclusion:

q: My theme is due.

Tautology:

$[p \wedge (p \rightarrow q)] \rightarrow q$

5. Hypothesis

p: Clara is a clerk in Meyers Dept. store

and $p \rightarrow q$: If a person is a clerk in Meyr's Dept. store, then he has had a week of on the job training.

Conclusion

q: Clara had a week of on the job training.

Tautology

$[p \wedge (p \rightarrow q)] \rightarrow q$

6. Hypothesis

p: Triangle ABC is equilateral

and $p \rightarrow q$: If a triangle is equilateral, then it is equiangular.

Conclusion

q: Triangle ABC is equiangular.

Tautology

$[p \wedge (p \rightarrow q)] \rightarrow q$.

7. Hypothesis

p: $4x=20$.

and $p \rightarrow q$: If equal numbers are divided by 4, the quotients are equal.

Conclusion

$q \rightarrow x=5$

Tautology

$[p \wedge (p \rightarrow q)] \rightarrow q$

8. Hypothesis

$p \rightarrow q$: If you can do this you're very bright.

$q \rightarrow r$: you're very bright, you'll probably get an "A" in logic.

Conclusion

$p \rightarrow r$: If you can do this you'll probably get an "A" in logic.

Tautology:

$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$

9. Hypothesis

$p \rightarrow q$: If an elephant is a rogue elephant, it is vicious.

$\wedge\ q \rightarrow r$: If an elephant is vicious, it is ostracized by the herd.

$\wedge\ r \rightarrow s$: If an elephant is ostrancized by the herd, it roams alone.

Conclusion

$p \rightarrow s$: If an elephant is a rogue elephant, it roams alone.

Tautology:

$[(p \rightarrow q) \wedge (q \rightarrow r) \wedge (r \rightarrow s)] \rightarrow (p \rightarrow s)$

10. The argument is valid.

Hypothesis:

Jerry is a veteran

If a man is a veteran, he's entitled to free medical care (Contrapositive)

Conclusion:

Jerry Anderson is entitled to free medical care.

11. The argument is valid.

Hypothesis.

All cows eat green grass.

If all cows eat gree grass, then all cows give rich milk (Contrapositive)

If cows give rich milk, then rain falls in summer (contrapositive).

Conclusion:

Rain falls in summer.

12. Hypothesis:

If California is not larger than Persia, then Utah is not larger than Maine. (Contrapositive)

If Utah is not larger than Maine, France is smaller than Texas.

If France is smaller than Texas, U.S. is not larger than Canada.

Conclusion:

If California is not larger than Persia, then the U. S. is not larger than Canada.

1.

Statements	Reasons
1. Jane lives in Denver	1. Given in Hypothesis
2. Jane lives in Colorado	2. If a person lives in Denver, then he lives in Colorado.

2.

Statements	Reasons
1. Tony is a student who lives more than a mile from school.	1. Given a Hypothesis.
2. Tony rides the bus.	2. If a student lives more than a mile from school, then he rides the school bus.
3. Tony eats lunch at school.	3. If a student rides the bus, then he eats lunch at school.

3.

Statements	Reasons
1. It is raining.	1. Given in Hypothesis
2. Pavements are slick.	2. If it rains, then pavements are slick.
3. There are more accidents than usual.	3. If pavements are slick, then there are more accidents than usual.

Proof by Contrapositive

Exercise Answers L46 - L47

1. Hypothesis: as given

   Conclusion: If an animal is a ku-ku, then it has grey fur.

   Contrapositive must be true.

2. Hypothesis: as given

   Conclusion: If a geometric figure is a triangle, then its sides are straight lines.

   Contrapositive must be true.

3. If I don't get algebra credit, then I didn't pass the final and turn in homework I missed.

   If I don't get algebra credit, then I didn't pass the final or I didn't turn in the homework I missed.

4. Argument is not valid.
5. Argument is not valid.
6. Argument is valid.
7. Argument is not valid.
8. Argument is not valid.

Exercise Answers - Indirect Proof

1. Assume all people in the room were born in different months. Then there are 13 different months. This contradicts the fact that only 12 months exist in the year. Therefore, at least two people in the room were born in the same month.
2. Assume each tree in Illinois has a different number of leaves on it. Let N be the number of trees in Illinois. Name each tree by the number of leaves it has. If no two trees have the same name, then one tree must be named for a number which is greater than or equal to N. This contradicts the hypothesis. Therefore, at least two trees have the same number of leaves.

- Exercise Answers -

Indirect Proof

3. Assume that there are an odd number of feet in the pen. Then at least one animal must have an odd number of feet. Any cow or chicken with an odd number of feet is not normal. Therefore, there cannot be an even number of feet in the pen.

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The Building Blocks of Logic

Activity Card I

This true-false test is designed to help you discover the real you. You may not speak to anyone during the test. Answer each question about yourself as candidly as possible.

___1. You have a well developed sense of humor.

___2. You do not have blue eyes.

___3. You passed Algebra with a C or better.

___4. If you are a student in this school then you graduated from it last June.

___5. You are not now and never have been a member of the faculty.

___6. You are an honest person.

___7. You have blue eyes and black hair.

___8. You have a winning personality.

___9. If you graduated from this school last June, then you are now taking geometry.

___10. You are generous to a fault.

___11. You have blue eyes.

___12. You are sensitive and understanding.

___13. You are tall or you have long hair.

___14. Don't fail this test.

___15. You have not spent one night in jail.

___16. You are not tall or you have long hair.

___17. Your smile is a loser or your smile is a loser only if you don't brush with dial.

___18. If a person has common sense then he is also logical.

___19. You are a student at this school.

___20. You have a lot of common sense.

Test Interpretation (Learn about yourself)

1. If you answered 2 and 11 the same you are illogical or you have one blue eye
2. If you answered 1,8,10,12 all true, then your answer for 6 should definitely be false.
3. If you answered 9 false you desperately need to study Logic. Luckily that's what you're about to do.
4. If you answered 15 true then how many nights did you spend in jail?
5. If your answers for 13 and 16 are both true, your hair is long or answer to 6 should be false.
6. If you answered 14 true you failed this test.
7. If your answer to 4,9, and 19 are all true, then you are doing post graduate work in geometry.
8. If your answer to 3 and 5 are false, then your answer to 15 is probably false now or will be false within the next week.
9. If your answer to 18 and 20 were true and your answer to 17 was false, you tend to be bashful and hide (successfully) your common sense.
10. You have enough potential to be allowed to proceed with the section on logic regardless of how you did on the test.

Exercises

Logic is a set of rules that we apply to arrive at valid conclusions.

A sentence is a statement if it is either true or false.

Example: $4 + 3 = 7$ True (a statement)

$4 + 2 = 5$ false (a statement)

Close the door. (not a statement)

You're late. True or False (a statement)

Write yes if the sentence is a statement, no if it is not.

——1. That car is speeding.

——2. Don't be late to class.

——3. $4 \times 3 = 6 \times 2$

——4. 13 is a prime number

——5. $6 + 3 = 7 + 3$

——6. $4 + y = 9$ if $y \in R$ (R represents the set of real numbers).

——7. Five is an even number.

——8. Add 4 and 10.

——9. $x + 2 = 10$ if $x \in$ (odd numbers)

——10. All even numbers are real numbers.

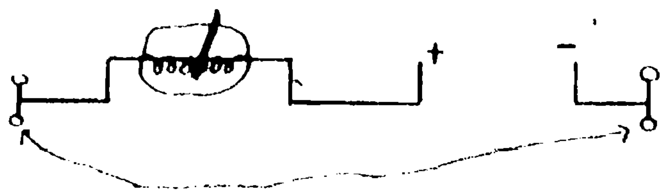
——11. Why don't you close the door?

——12. A statement is a sentence that is either true or false.

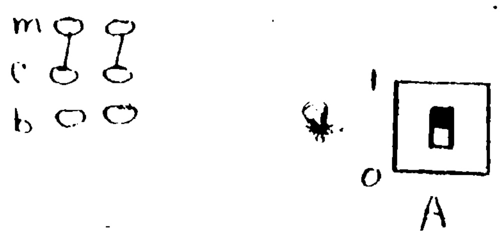
The Circuit Board

Electric current flows from + to - if there is a path provided for its flow. Any continuous wire connection between a + and - terminal will provide a pathway.

1. On your circuit board see if you can turn a light on by making a direct path from + to - for the current.



To avoid having to wire a circuit whenever one wishes a light on and disconnecting it when no light is desired, switches are used. The switches on your circuit board operate as shown in the diagrams. When switch A is at position 1 there is a direct path from m to c.



When switch A is at the o position there is a direct path from c to b.

2. Place a wire from the positive terminal to an m eyelet and another from the negative terminal to the corresponding c eyelet. The light has value 1 if it is on and o if it is off.

   If the value of A is 1, the value of the light is ____.

   If the value of A is o, the value of the light is ____.

Arrange these results in a table:

A	Light
1	
0	

(Table 1)

The light is equivalent to switch A or the light represents swith A.

3. Place a wire from the positive terminal to a b eyelet and from the negative terminal to the corresponding c eyelet.

Complete the table:

A	Light
1	
0	

(Table 2)

The light is not equivalent to switch A. In fact it has exactly the opposite value. We say the light represents "not A".

Exercises

If two statements are such that when one is true the other is true, and when one is false the other is false, they are called equivalent statements. We can show this definition in a table. Let p represent the first statement and q represent the second statement. The symbols T and F stand for true and false.

p	q	
T	T	Both statements are true.
F	F	Both statements are false.

If you have done activity II with circuit boards this table should look familiar. Compare it to Table I. They are really the same table. Only the symbols have been changed. Both are called truth tables.

We use the symbol $\longleftrightarrow$ to indicate equivalence. $p \longleftrightarrow q$ is read "p is equivalent to q". Note these properties of equivalence:

$p \longleftrightarrow q$ (Equivalence is reflexive)

If $q \longleftrightarrow q$, then $q \longleftrightarrow p$ (Equivalence is symmetric)

If $p \longleftrightarrow q$, $q \longleftrightarrow r$, then $p \longleftrightarrow r$ (Equivalence is transitive)

Beside each of the following pairs of algebra statements write yes if they are equivalent and no if they are not.

1. $2x + 1 \quad 17$
   $2x = 16$
2. $x > 7$
   $x \leq 7$
3. $3y \leq 18$
   $y \leq 6$
4. $5x \neq 15$
   $x = 3$
5. $x \leq 5$
   $x \neq 5$
6. $\frac{1}{3}x = \frac{2}{3}$
   $x = 2$

7. In the above exercises, two algebraic statements are equivalent if they have the

same ____________________.

8. Complete this truth table for $p \leftrightarrow q \leftrightarrow r$.

p	q	r
T		
F		

a) what do you conclude about p and r?

b) What property of equivalence is illustrated?

Beside the following pairs of statements write yes if they are equivalent and no if they are not.

9. It is January.

It is not February.

10. It is not cold in here.

It is false that it's cold in here.

11. It is raining

It is not raining

12. Today is Sunday.

Today is the first day of the week.

13. Today is Christmas.

Today is a holiday.

14. If you work hard, then you will be a success.

If you are a success, then you worked hard.

Exercises

Two statements that never have the same truth values are said to be negations . That is they are never both true at the same time nor both false at the same time.

From the following list of statements select any pairs you think are negations.

p. Today is Monday

q. Today is Tuesday

r. Today is Wednesday

s. Today is not Monday

t. p is false.

Are statements p and q negations? Remember they could both be false (on Wed. for example) To be negations they must always have opposite truth values.

If you selected p and s as negations you made a correct choice. The truth table would look like this

p	s	
T	F	(when p is true, s is false)
F	T	(when p is false, s is true).

We designate the negation of "p" by "$\sim$p" which is read "not p" or "negation of p"..

Since S is a negation of p we could replace s in the truth table with $\sim$p:

P	$\sim$ P
T	F
F	T

If you did activity II with circuit boards compare this truth table with table 2. Again they are the same truth table.

1. Complete the truth table.

P	$\sim$ P	$\sim(\sim$P)
T		
F		

a) What do you notice about p and $\sim(\sim p)$? They are ______________ statements.

2. Write a negation of each of the following statements.

Example: Statement $y > 7$

Negation y is not > 7

Better negation $y \leq 7$

a) $y \leq 5$

b) $x + 4 = 9$

c) $2y + 4 = 8$

d) $x \neq 9$

e) $y < 5$

f) $3y \geq y + 5$

3. Complete this truth table showing q is a negation of p, r is a negation of q, s is a negation of r.

P	Q	R	S
T			
F			

a) What do you conclude about p and s?

4. Complete the table when r is a negation of p and s is a negation of q.

P	Q	R	S
T	T		
F	F		

a) What do you conclude about p and q?

b) What do you conclude about r and s?

c) Negations of equivalent statements are ______________.

5. If p is the statement, $x < 2$, what is $\sim[\sim(\sim p)]$?

Let's look again at our list of statements from which we selected pairs of negations.

p: Today is Monday.

q. Today is Tuesday

r. Today is Wednesday

s. Today is not Monday

t. p is false

You probably noticed another pair of negations besides p and s. Consider the truth table for p and t.

p	t
T	F
F	T

(when p is true, t must be false)

(when p is false, t is true).

There is more than one way to write a negation of a statement.

Example:

p: Everyone in this class is brilliant.

p: Not everyone in this class is brilliant

p: It is false that everyone in this class is brilliant.

To make p false we need to find only one person in the the class that is not brilliant. We could write: p: One person in this class is not brilliant. Of course if we find more than one non-brilliant person p is just as false.

$\sim$p: Some people in this class are not brilliant.

Write three different negations of each of the following:

6. All of us are hungry.
7. Some people sleep in class.
8. No student likes Geometry.
9. Richard is never serious.
10. None of us will be here tomorrow.

Review

1. The light is equivalent to switch A.

A	Light
1	1
0	0

A wire connects the positive to a m eyelet of switch A, and another connects the negative with corresponding c eyelet.

We will use the following notation in circuit diagrams:

2. The light is the negation of switch A.

A	Light
1	0
0	1

A wire connects the positive to a b eyelet of switch A, and another connects the negative with the corresponding c eyelet.

We will use the following notation in circuit diagrams:

Wire the following circuits and complete the table. Work with two switches and one light.

1.

A	B	Light
1	1	
1	0	
0	1	
0	0	

And Circuit

Electricity must go through A and B to the light.

2.

A	B	Light
1	1	
1	0	
0	1	
0	0	

Or Circuit Electricity can go through A or B to the light.

Exercises

P and q be any logical statements., The compound statement "p and q" is called the conjunction of p and q and is written $p \vee q$.

Example: p: Jack is a tennis buff.

q: Jack plays basketball.

$p \vee q$: Jack is a tennis buff and Jack plays basketball.

Under what conditions is a conjunction true? For instance suppose p is true but Jack really doesn't play basketball making q false. Is the conjunction true or false? Enter your answer in the second row of this truth table. Complete the table.

1.

p	q	$p \wedge q$
T	T	
T	F	
F	T	
F	F	

If you did Activity III compare this table to the table for an And circuit.

2. A conjunction of two statements is true if and only if ____________.

The compound statement "p or q" is called a disjunction and is written "$p \vee q$".

Example

p: John likes math.

q: John likes history.

$p \vee q$: John likes math or John likes history.

Suppose it's true that John likes math but he really doesn't like history. Is $p \vee q$ true or false in this case? Enter your answer in the second row of this truth table and complete the table.

3.

P	Q	$P \vee Q$
T	T	
T	F	
F	T	
F	F	

Exercises

If you did Activity III compare this table to the table for an Or circuit.

4. A disjunction of two statements is false if and only if ______________.

5. p: Mastedons were big

   q: Men don't walk up walls.

a) Write $p \wedge q$:

b) Write $p \vee q$:

6. p: Every triangle has three sides

   q: Every square has four right angles.

a) Write $p \wedge q$:

b) Write $p \vee q$:

c) Is $p \wedge q$ true or false?

d) Is $p \vee q$ true or false?

7. p: a pentagon has five sides

   q: A square has three diagonals.

a) Write $p \wedge q$:

b) Write $p \vee q$:

c) Is $p \wedge q$ true or false?

d) Is $p \vee q$ true or false?

8. A conjunction $r \wedge s$ is true if ____.

a) R is true and s is false.

b) r is true and s is true

c) r is false and s is true.

9. If "Wolligogs eat fish and Gilliwogs drink Bo" is a true statement, then the statement "Gilliwogs drink Bo"_____.

a) may be true b) must be false c) must be true

10. Suppose p is true and q is true.

a) Is $\sim p$ true or false?

b) Is $\sim p \wedge q$ true or false?

11. r is true and s is true. State whether each of the following is true or false.

a) _____ $r \wedge s$

b) _____ $r \wedge \sim s$

c) _____ $\sim r \wedge s$

d) _____ $\sim r \vee \sim s$

e) _____ $\sim r \wedge \sim s$

f) _____ $r \wedge \sim s$

More Complex Circuits

Activity Card 4

Review

Logic Statement

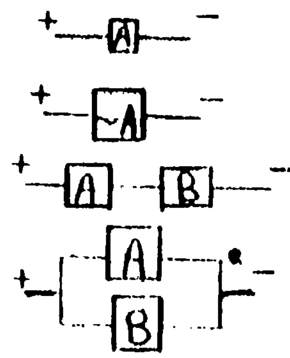
A

~A

A∧B

A∨B

Circuit Diagram



More complicated statements can be built from these basic statements

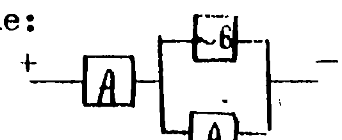
More complicated circuits can be built from these basic circuits.

Step I: You will be given a logical statement

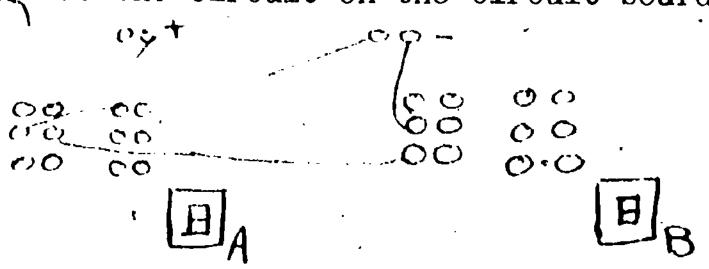
Example $A \wedge (\sim B \vee A)$

Step II: You will draw a circuit diagram for the statement.

Example:



Step III: Wire the circuit on the circuit board.



Step IV: Complete the truth table for this statement.

A	B	A∧(~B∨A)
1	1	1
1	0	1
0	1	0
0	0	0

New Complex Circuits

Review

Follow these four steps for each of the following.

1. $\sim A \wedge \sim B$

Circuit:

A	B	$\sim A \wedge \sim B$
1	1	
1	0	
0	1	
0	0	

2. $\sim A \vee \sim B$

Circuit:

A	B	$\sim A \wedge \sim B$
1	1	
1	0	
0	1	
0	0	

3. $A \vee (\sim B \wedge A)$

Circuit:

A	B	$A \vee (\sim B \wedge A)$
1	1	
1	0	
0	1	
0	0	

4. $(A \wedge \sim B) \wedge A$

Circuit:

A	B	$(A \vee \sim B) \wedge A$
1	1	
1	0	
0	1	
0	0	

5. $(A \wedge B) \vee (\sim A \wedge \sim B)$

Circuit:

A	B	$(A \wedge B) \vee (\sim A \wedge \sim B)$
1	1	
1	0	
0	1	
0	0	

6. $(A \vee B) \wedge (\sim A \vee \sim B)$

Circuit:

A	B.	$(A \vee B) \wedge (\sim A \vee \sim B)$
1	1	
1	0	
0	1	
0	0	

By now you are familiar with the idea of a truth table and ready to complete more complicated ones. Let's take this compound statement,

$$p \wedge (\sim q \vee p)$$

and develop a truth table

1. Determine how many simple statements are in the compound statement. This example has two: p and q. We will need a column heading for each.

p	q

2) For these two statements there are four possible situations: 1) both are true, 2) p is true, q is false, 3) p is false, q is true, 4) both are false.

p	q
T	T
T	F
F	T
F	F

The table will require four rows.

3. Make a column for any negations of simple statements.

p	q	~q
T	T	F
T	F	T
F	T	F
F	F	T

Opposite truth value

Exercises

Compound Statements and DeMorgan's Laws

4. Determine truth value of parentheses

use these two columns

p	q	~q	(~q ∨ p)
T	T	F	T
T	F	T	T
F	T	F	F
F	F	T	T

5.

use these two columns

p	q	~q	(~q ∨ p)	P ∧ (~q ∨ p)
T	T	F	T	T
T	F	T	T	T
F	T	F	F	F
F	F	T	T	F

Construct a truth table for each statement.

1. ~(~p ∨ ~ q)

p	q	~p	~ q	(~p ∨ ~q)	~(~p ∨ ~q)

2. p ∨ ~ q

3. P ∧ ~q

4. ~p ∨ q

5. ~p ∨ ~q

6. (p ∨ ~q) ∧ (q ∨ ~p)

60

7. $\sim(p \vee q)$

8. $\sim p \wedge \sim q$

Notice these two truth tables have the same last column When one statement is true, the other is true, when one is false the other is false. They are equivalent

$\sim(q \vee p) \leftrightarrow \sim p \wedge \sim q$

9. $\sim(p \wedge q)$

10. $\sim p \vee \sim q$

Notice again $\sim(p \wedge q) \leftrightarrow \sim p \vee \sim q$

These equivalences are called DeMorgan's Laws

11. Write the appropriate statements for:

p: Jim will work

q: Bill will work

$\sim(p \wedge q)$: It is false that Jim and bill will work.

$\sim p \vee \sim q$:

12. Write the appropriate statements for:

p: Joe graduates in June

q: Jerry will be a Junior

$\sim(p \wedge q)$:

$\sim p \vee \sim q$:

13. Let's construct a truth table for a compound statement of three statements: p, q, r. How many different situations would there be?

p	q	r	$p \wedge q$	$(p \wedge q) \vee r$

Each of the following activities will require:

a) players (students to act out small parts)

b) a recorder (student to keep records visible to whole class.)

c) The unsilent majority (remainder of the class which will react favorably or unfavorably to each situation).

In each act one of the players makes a conditional commitment to another. In each scene he will either break his promise (greeted by boos from the unsilent majority) or not break his promise (yea, he's true blue). The recorder will take down the results so they may be studied later.

Record

Act: 1: Charley Makes A Date

Scene	Charley gets $5 from Dad: (true or false)	Charley takes Lucy to movies (true or false)	Charley's promise broken (yea or boo)
1			
2			
3			
4			

Act 2: Charley and the Teacher

Scene	Charley stays out of trouble (true or false)	Charley passes (true or false)	Ms. Shipley's promise broken (yea or boo)
1			
2			
3			
4			

Act 3: Charley's Ski Trip

Scene	Charley doesn't pass geometry (true or false)	Brown's take Charley skiing (true or false)	Ms. Brown's promise broken? (yea or boo)
1			
2			
3			
4			

Act 1: Charley Makes a Date

Characters: Charley (a high school student), Lucy (Charley's girl), Mr. Brown (Charley's dad).

Situation: Charley makes the following promise to Lucy. "If I can get five dollars from dad, then I'll take you to the movies Friday night."

Scene 1: Charley makes the promise

Mr. Brown gives Charley the money.

Charley and Lucy go to the movies.

Class reacts to the question "Did Charley break his promise?"

Record the reaction.

Scene 2: Charley makes the promise.

Mr. Brown gives Charley the money.

Charley tells Lucy he can't take her to the movies.

Class reacts: "Did Charley break his promise?"

Record reaction.

Scene 3: Charley makes the promise

Mr. Brown does not give Charley the money.

Charley and Lucy go to the movies.

Class reacts: "Did Charley break his promise?"

Record reaction.

Scene 4: Charley makes the promise.

Mr. Brown does not give Charley the money.

Charley tells Lucy he can't take her to the movies.

Class reacts "Did Charley break his promise?"

Record reaction.

Act 2: Charley and the Teacher

Characters: Charley, Ms. Shapley (Charley's Geometry teacher), Linus, Lucy, Snoopy (Charley's classmates).

Situation: Ms. Shapley makes the following promise to Charley, "if you stay out of trouble in class, then you'll get a passing grade."

Scene 1: Ms. Shapley makes the promise.

Charley stays out of trouble.

Ms. Shapley gives Charley a passing grade.

Class reacts "Did Ms. Shapley break her promise?"

Record reaction

Scene 2: Ms. Shapley makes the promise.

Charley stays out of trouble.

Ms. Shapley gives Charley a failing grade.

Class reacts "Did Ms. Shapley break her promise?"

Record reaction.

Scene 3: Ms. Shapley makes the promise

Charley gets into trouble.

Ms. Shapley gives Charley a passing grade.

Class reacts "Did Ms. Shapley break her promise?"

Record reaction.

Scene 4: Ms. Shapley makes the promise.

Charley gets into trouble.

Ms. Shapley gives Charley a failing grade.

class reacts, "Did Ms. Shapley break her promise?"

Record reaction.

Act 3: Charley's Ski Trip

Characters: Charley, Mrs. Brown (Charley's mother) Ms. Shapley.

Situation: Mrs. Brown tells Charley the following: "you will not pass Geometry or we will take you skiing."

Scene 1: Mrs. Brown makes the promise.

Ms. Shapley gives Charley a passing grade.

Mrs. Brown does let Charley go skiing.

Class reacts "Did Mrs. Brown keep her promise?

Record reaction

Scene 2: Mrs. Brown makes the promise.

Ms. Shapley gives Charley a passing grade.

Mrs. Brown doesn't let Charley go skiing.

Class reacts "Did Mrs. Brown keep her promise.

Record reaction

Scene 3: Mrs. Brown makes the promise.

Ms. Shapley doesn't give Charley a passing grade.

Mrs. Brown lets Charley go skiing.

Class reacts "Did Mrs. Brown keep her promise?"

Record reaction.

Scene 4: Mrs. Brown makes the promise.

Ms. Shapley doesn't give Charley a passing grade.

Mrs. Brown doesn't let Charley go skiing.

Class reacts "Did Mrs. Brown keep her promise."

Record reaction.

Exercises

1. Complete the truth table for p q

p	q	$\sim p$	$\sim p \vee q$
T	T		
T	F		
F	T		
F	F		

Let p: You go to Europe

q: You cross an ocean.

Then $\sim p \vee q$: You don't go to Europe or you cross the ocean.

Looking at the truth table above, this statement is false only when you go to Europe and don't cross an ocean. (which is not likely).

It would be equivalent to say:

If you go to Europe, then you cross an ocean. A statement like this is called an if-then statement, or an implication. Symbolically we write $p \rightarrow q$ which is read "if p, then q" or "p implies q."

Since "$\sim p \vee q$" and "$p \rightarrow q$" are equivalent the truth table for $p \rightarrow q$ should have the same result as the one you completed above.

2. Complete the truth table:

p	q	$p \rightarrow q$
T	T.	
T	F	
F	T	
F	F	

If you did Activity 6 compare this table to table 1.

There are many ways to write an implication $p \rightarrow q$.

If p, then q. If it's raining, then it's cloudy.

p implied q. It's raining implies it's cloudy.

Implications

Exercises

q if p	It's cloudy if it's raining.
p only if q	It's raining only if it's cloudy.
p is sufficient for q	It's raining is suffficient for it's cloudy.
q is necessary for p	It's cloudy is necessary for it's raining.

In each of these examples p is called the hypothesis and q is the conclusion

Hypothesis: It'is raining

conclusion: It is cloudy

Underline the hypothesis once and the conclusion twice:

3. If it rains then we'll go home early.
4. If we win there will be no school tomorrow.
5. $4x=20$ implies $x=5$.
6. We're going only if the weathers nice.
7. To be eligible it's necessary to keep your grades up.

write the following in if-then form.

8. If I eat less, I lose weight.
9. If the cat's away, the mouse will play.
10. I'll be there if I finish my homework in time.
11. You like a course only if you do well in it.
12. All teachers must attend the meeting.
13. To join the club it is necessary to be a student.
14. Being a cheer leader implies being a junior.
15. You live in Colorado if you live in Denver.
16. A triangle has three sides.
17. Living in Denver is sufficient for living in Colorado.
18. Living in Colorado is necessary for living in Denver.

Consider the statement:

It is not raining or it is cloudy.

This statement is false only if it is raining and there are no clouds, a very unlikely possibility.

It appears that an equivalent statement would be:

If it rains, then it is cloudy.

or

Rain implied clouds.

Both of these statements are implications. Symbolically, if p: It is raining., and q: It is cloudy we would write

$p \longrightarrow q$

It seems from our example that $p \rightarrow q$ is equivalent to $\sim p \vee q$. In fact we shall define it that way. Definiition: $p \longrightarrow q$ is a statement equivalent to $\sim p \vee q$.

Use a circuit board to find the truth table for $A \longrightarrow B$.

A	B	A $\longrightarrow$ B
1	1	
1	0	
0	1	
0	0	

Table 1

Converse

The converse of $A \longrightarrow B$ is $B \longrightarrow A$.

This is equivalent to $\sim B \vee A$.

Use a circuit board to complete the table.

A	B	B $\longrightarrow$ A
1	1	
1	0	
0	1	
0	0	

Table 2

Inverse

The inverse of $A \longrightarrow B$ is $\sim A \longrightarrow \sim B$

This is equivalent to $A \vee \sim B$.

Use a circuit board to complete the table.

A	B	$\sim A \longrightarrow \sim B$
1	1	
1	0	
0	1	
0	0	

Table 3

Contrapositive

The contrapositive of $A \longrightarrow B$ is $\sim B \longrightarrow \sim A$.

This is equivalent to $B \vee \sim A$.

Use a circuit board to complete the table.

A	B	$\sim B \longrightarrow \sim A$
1	1	
1	0	
0	1	
0	0	

Table 4

The converse of $p \rightarrow q$ is $q \rightarrow p$

Example: $p \rightarrow q$ if you're smart, then you go to college.

$q \rightarrow p$ if you go to college, then you're smart.

Go back to Implication Exercises 8 - 11 and write the converse of each.

1.

2.

3.

4.

5. Complete the truth table

p	q	$p \rightarrow q$	$q \rightarrow p$
T	T		
T	F		
F	T		
F	F		

If you did Activity 6 compare this table to table 2.

6. When an implication is true, is its converse true?

7. Make up an implication that's true and has a true converse.

8. Make up an implication that's true and has a false converse.

A conjunction (and) of an implication and its converse is called a biconditional.

Implication $p \rightarrow q$: If it's Monday, then tomorrow is Tuesday.

Converse $q \rightarrow p$: If tomorrow is Tuesday, then it's Monday.

Biconditional $(p \rightarrow q)$ $(q \rightarrow p)$: If it's Monday, then tomorrow is Tuesday and if tomorrow is Tuesday, then it's Monday.

9. Complete the truth table.

p	q	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \wedge (q \rightarrow p)$
T	T			
T	F			
F	T			
F	F			

There are several other simpler ways to state a biconditional. Notice in the truth table the biconditional is true only when p and q have same truth value and false when p and q have different truth values. We can state the biconditional simply p "is equivalent to" q or $p \leftrightarrow q$.

Example: It's Monday is equivalent to tomorrow is Tuesday.

Another way to write a biconditional statement is to combine if p, then q and p only if q to p if and only if q.

Example: Today is Monday is neccesary and sufficient for tomorrow being Tuesday.

A third way is to combine:

p is sufficient for q and p is necessary for q

to

p is necessary and sufficient for q.

Example: Today is Monday is necessary and sufficient for tomorrow being Tuesday.

Write each biconditional statement in two other ways.

I can wake up if and only if I go to sleep.

10.

11.

A polygon has three sides is equivalent to a polygon is a triangle.

12.

13.

I live in Denver is necessary and sufficient for I live in the Mile High City.

14.

15.

Exercises

The inverse of $p \rightarrow q$ is $\sim p \rightarrow \sim q$.

Example: $p \rightarrow q$ If you live in Denver, then you live in Colorado.

$\sim p \rightarrow q$ If you don't live in Denver, then you don't live in Colorado.

Go back to Implication Exercises 12–15 and write the inverse of each.

1.

2.

3.

4.

5.

Complete the table:

p	q	$p \rightarrow q$	$\sim p$	$\sim q$	$\sim p \rightarrow \sim q$
T	T				
T	F				
F	T				
F	F				

If you did Activity 6 compare this table to table 3.

6. If an implication is true, is it's inverse always true?

7. Make up an implication that's true, and has a true inverse.

8. Make up an implication that's true, and has a false inverse.

Exercises

The contrapositive of $p \rightarrow q$ is $\sim q \rightarrow \sim p$.

Example: $p \rightarrow q$: If you finished the test, then go to the computer room.

$\sim q \rightarrow \sim p$: If you don't go to the computer room then you didn't finish The test.

Go back to exercises 16-17 and write the contrapositive of each.

1.

2.

3. Complete the table

p	q	$p \rightarrow q$	$\sim p$	$\sim q$	$\sim q \rightarrow \sim p$
T	T				
T	F				
F	T				
F	F				

If you did Activity 6 compare this to table 4.

4. If an implication is true, is it's contrapositive always true?

5. An implication and it's contrapositive are ________ statements.

Draw a circuit for $A \rightarrow (A \vee B)$

Use a circuit board to complete the following table.

A	B	$A \rightarrow (A \vee B)$
1	1	
1	0	
0	1	
0	0	

Table 1

Draw a circuit for $B \wedge (A \wedge \sim B)$

A	B	$B \wedge (A \wedge \sim B)$

Table 2

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Complete the truth table:

1.

p	q	$(p \vee q)$	$p \rightarrow (p \vee q)$
T	T		
T	F		
F	T		
F	F		

If you did Activity 7 compare this to table 1.

A statement like this one that is always true is called a tautology.

Complete the truth table.

2.

p	q	$\sim q$	$(p \wedge \sim q)$	$q \wedge (p \wedge \sim q)$
T	T			
T	F			
F	T			
F	F			

If you did Activity 7 compare this to table 2.

A statement like this one that is always false is called a contradiction.

Complete the following truth tables and state whether each is a tautology, contradiction or neither.

3. $p \vee \sim p$

p	
T	
F	

4. $p \wedge \sim p$

p	
T	
F	

5. $(p \wedge q) \longrightarrow p$

p	q	
T	T	
T	F	
F	T	
F	F	

6. $p \longrightarrow (p \wedge q)$

T	T	
T	F	
F	T	
F	F	

Exercises

7. $\sim(r \vee s) \longleftrightarrow \sim r \wedge \sim s$

r	s	
T	T	
T	F	
F	T	
F	F	

8. $[p \wedge (p \rightarrow q)] \rightarrow q$

p	q	
T	T	
T	F	
F	T	
F	F	

9. $\sim(p \wedge \sim q) \vee (p \wedge q)$

p	q	
T	T	
T	F	
F	T	
F	F	

10. $(p \vee \sim p) \rightarrow (q \wedge \sim q)$

p	q	
T	T	
T	F	
F	T	
F	F	

11. $(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p)$

p	q	
T	T	
T	F	
F	T	
F	F	

12. $(p \wedge \sim q) \vee (p \wedge q)$

p	q	
T	T	
T	F	
F	T	
F	F	

13. $[(p \rightarrow q) \cap (q \rightarrow r)] \rightarrow (p \rightarrow r)$

p	q	r	
T	T	T	
T	T	F	
T	F	T	
T	F	F	
F	T	T	
F	T	F	
F	F	T	
F	F	F	

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In his book Symbolic Logic, Lewis Carroll presents sets of concrete propositions and asks what conclusions may be drawn from them. In this activity we present selected sets of these along with conclusions. You are asked to determine if those conclusions are valid or not. It it's not valid, write one that is.

I.

1. Babies are illogical.
2. Nobody is despised who can manage a crocodile.
3. Illogical persons are despised.

Concl: Babies can manage crocodiles.

II.

1. No potatoes of mine, that are new, have been boiled.
2. All my potatoes in this dish are fit to eat.
3. No unboiled potatoes of mine are fit to eat.

Concl: All potatoes in this dish are old ones.

III.

1. Everyone who is sane can do Logic.
2. No lunatics are fit to serve on a jury.
3. None of our students can do logic.

Concl: None of our students are fit to serve on a jury.

IV.

1. No one takes in the Post, unless he is well-educated.
2. No marmot can read.
3. Those who cannot read are not well-educated.

Concl: Some marmots take in the Post.

V.

1. All ducks in this village that are branded "B" belong to Mrs. Bond.
2. Ducks in this village never wear lace collars, unless they are branded "B".

3. Mrs. Bond has no gray ducks in this village.

Concl: All white ducks in this village wear lace collars.

VI.

1. Nobody, who really appreciated Beethoven, Fails to keep silence while the Moonlight-Sonata is being played.

2. Guinea - pigs are hopelessly ignorant of music.

3. No one, who is hopelessly ignorant of music ever keeps silence while the Moonlight - Sonata is being played.

Concl: Guinea pigs never really appreciate Beethoven.

VII.

1. Things sold in the street are of no great value.

2. Nothing but rubbish can be had for a song.

3. Eggs of the Great Auk are very valuable.

4. It is only what is sold in the streets that is really rubbish.

Concl: Those things which are sold on the streets and are not rubbish are eggs of the Great Auk.

VIII.

1. No birds, except ostriches, are 9 feet high.

2. There are no birds in this aviary that belong to anyone but me.

3. No ostrich lives on mince - pies.

4. I have no birds less than 9 feet high.

Concl: Any bird not living on mince pies is in this aviary.

IX.

1. All writers who understand human nature, are clever.

2. No one is a true poet unless he can stir the hearts of men

3. Shakespeare wrote "Hamlet".

4. No writer, who does not understand human nature can stir the hearts of men.

Would You Believe?

5. None but a true poet could have written "Hamlet".

Concl: Shakespeare was clever.

X.

1. I call no day "unlucky" when my students are civil to me.
2. Wednesdays are always cloudy.
3. When people take umbrellas, the day never turns out fine.
4. The only days when my students are uncivil to me are Wednesdays.
5. Everyone takes his umbrella with him when it is raining.
6. My "lucky" days always turn out fine.

Concl: Draw your own conclusions for this one.

Exercises

Proofs

Looking back at the last exercises, these are some of the statements you found to be tautologies.

$$p \vee \sim p$$

$$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$$

$$[p \wedge (p \rightarrow q)] \rightarrow q$$

Any argument that follows a tautology is a valid argument.

For example, if you argue following the first tautology, "The sky is green or it is not green", you have to be right because $p \vee \sim p$ is always true.

1. On the second and third tautologies above underline the hypothesis once and the conclusion twice. You might recognize the second tautology on the list as the transitive property of implication. Here's an example of an argument following this tautology.

Hypothesis:

$(p \rightarrow q)$: If John lives in Golden, then he lives in Jefferson County.,

and $(q \rightarrow r)$; If John lives in Jefferson County, then he lives in Colorado.

Conclusion:

$(p \rightarrow r)$: If John lives in Golden, then he lives in Colorado.

The argument follows the tautology $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$

Example of an argument following the third tautology.

Hypothesis:

p: John is a football player.

and $(p \rightarrow q)$: All football players are dumb.

Conclusion:

q: John is dumb.

Tautology: $[p \wedge (p \rightarrow q)] \rightarrow q$

An argument following a tautology is valid because for everyone who agrees the hypothesis is true, the argument is a proof of the conclusion.

Hypothesis is:

p: A bicycle is a vehicle and $(p \rightarrow q)$: All vehicles have four wheels.

Conclusion:

q: A bicycle has four wheels.

Tautology: $[p \wedge (p \rightarrow q)] \rightarrow q$

Notice the argument is valid because it follows a tautology, but the conclusion is false.

The following statements form the hypothesis of a valid argument. You are to arrange the statements in order, state the conclusion, and state the tautology the argument is patterned after. (See previous examples).

2. I am a freshman student. If a student is a freshman, then he may attend the freshman class party.

3. If I eat pickles with ice cream, I dream about lions and tigers. If I dream about lions and tigers, I wake up scared.

4. If this is the fourth week of school, then this is the week my theme is due. This is the fourth week of school.

5. If a person is a clerk in Meyer's Department store, then he has had a week of on the job training. Clara is a clerk in Meryer's Department Store.

6. Triangle ABC is equilateral. If a triangle is equilateral, then it is Equiangular.

7. $4X=20$. If equal numbers are divided by 4, the quotients are equal.

8. If you're very bright you'll probably get an A in logic. If you can do this you're very bright.

9. If an elephant is vicious, it is ostracized by the herd.
   If an elephant is a rogue elephant, it is vicious.
   If an elephant is ostracized by the herd, it roams alone.

Remember that an implication is equivalent to its contrapositive, so we can use the contrapositive in place of an implication, to make a valid argument. In the following arguments you are given the hypothesis and conclusion. You are to determine whether or not the argument is valid by determining whether or not it follows a tautology

Example:

Hypothesis

If the price of this coffee is high, then some people do not buy it.

All people buy this coffee.

Conclusion

The price of this coffee is not high.

The argument is valid:

Hypothesis

p: All people buy this coffee.

$p \to q$. If all people buy this coffee, then the price of this coffee is not high. (contrapositive of first statement)

Conclusion

q: The price of this coffee is not high.

10. Hypothesis:

If a man is not entitled to free medical care, he's not a veteran.

Jerry Anderson is a veteran.

Conclusion:

Jerry Anderson is entitled to free medical care.

11. Hypothesis:

All cows eat green grass.

If no cows give rich milk, then no cows eat green grass.

If no rain falls in summer, then no cow gives rich milk.

Conclusion

Rain falls in summer.

12. Hypothesis

Utah is larger than Maine only if California is larger than Persia. France is smaller than Texas if Utah is not larger than Maine. The fact that the U.S. is not larger than Canada is a necessary condition that France be smaller than Texas.

Conclusion:

(Find a conclusion)

Exercises

A popular form of proof in Geometry is to arrange an argument following the tautology $[p \wedge (p \rightarrow q)] \rightarrow q$ in two columns.

Statements	Reasons
1. p	1. Given in Hypothesis
2. q	2. $p \rightarrow q$

Example: Prove that John is dumb.

statements	Reasons
1. p: John is a football player	1. Given in Hypothesis
2. q: John is dumb.	2. $p \rightarrow q$: All football players are dumb.

Having accomplished this, knowing that q is true, we can use the argument $[q \wedge (q \rightarrow r)] \rightarrow r$.

Statements	Reasons
1. p	1. Given in Hypothesis.
2. q	2. $p \rightarrow q$
3. r	3. $q \rightarrow r$

But that means we have really made the argument:

$$[p \wedge (p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow r.$$

Example: Prove the price of bread will rise.

statements	Reasons
1. p: The cost of labor is increasing	1. Given in Hypothesis
2. q: The cost of wheat increases.	2. $p \rightarrow q$: If the cost of labor increases, then the cost of wheat increases.
3. R: The price of bread will rise.	3. $q \rightarrow r$: If the cost of wheat increases then the price of bread will rise.

Exercises

Arrange each of the following arguments in two column form.

1. Hypothesis: Jane lives in Denver. If a person lives in Denver, then he lives in Colorado.

   Conclusion (prove): Jane lives in Colorado.

Statements	Reasons

2. Hypothesis: If a student lives more than a mile from school, then he rides the school bus. If a student rides the bus, then he eats lunch at school. Tony is a student who lives more than a mile from school.

Conclusion (prove): Tony eats lunch at school.

Statements	Reasons

3. Hypothesis If it rains, then pavements are slick. If pavements are slick, then there are more accidents than usual. It is raining today.

   Conclusion (prove): Whatever you can.

Statements	Reasons

Proof by Contrapositive

Another tautology you proved by truth table:

$$(p \rightarrow q) \longleftrightarrow (\sim q \rightarrow \sim p)$$

An implication is equivalent to it's contrapositive.

Sometimes a true implication cannot be proved from the hypothesis given, but the contrapositive can be proven instead. This is still a valid proof of an implication since it is equivalent to its contrapositive.

Exercises: prove the contrapositive

1. Hypothesis

If an animal is a ku-ku, then it has three ears.

If an animal has three ears, then it has grey fur.

Prove: If an animal does not have grey fur, it is not a ku-ku.

2. Hypothesis:

If a geometric figure is a triangle, it is a polygon.

If a geometric figure is a polygon, then its sides are segments of straight lines.

Prove: If a geometric figure does not have its sides segments of straight lines, it is not a triangle.

3. Write two contrapositives: I shall get my algebra credit if I pass the final and turn in the homework I missed.

Determine whether or not the argument is valid.

4. Hypothesis: If all cows eat green grass, then some cows give rich milk.

If no rain falls in summer, then no cow gives rich milk.

Some rain falls in summer.

Conclusion: All cows eat green grass.

5. Hypothesis: If the student does not study hard, then he fails the course.

If the student studies hard, then he has no regrets.

The student fails the course.

Conclusion: The student has some regrets.

6. Hypothesis: If some mountains are not high, then some peaks are not snow capped.

If no skiing is done in Boulder, then some mountains are not high.

All peaks are snow capped

Conclusion: Some skiing is done in Boulder.

7. Hypothesis: If and only if potatoes are planted at full moon, they will make a heavy crop. If the potato crop is small, wheat prices are low.

Conclusion: If potatoes are planted at full moon, wheat prices will be high.

8. Hypothesis: Eating a green apple is sufficient to cause illness.

You call a doctor if your life is in danger.

To call a doctor, it is necessary to be ill.

Conclusion: If you eat a green apple, your life is in danger.

Indirect Proof

Exercises

Negation

Sometimes in order to prove that a statement is true, we prove that it is false. This method is known as indirect proof.

Suppose q is a statemnt we wish to prove. Assume q is true, and reason deductively until you arrive at a statement t which you know is false. In other words, construct a deductive sequence from q to t as follows:

$\sim q \rightarrow a \rightarrow b \rightarrow \ldots t$. This sequence establishes the implication $\sim q \rightarrow t$. The contrapositive of $\sim q \rightarrow t$ is $\sim t \rightarrow \sim(\sim q)$. Since t is false, $\sim t$ is true and $\sim(\sim q) \leftrightarrow q$. You may illustrate your reasoning by

$$\left.\begin{matrix}\sim t \\ \sim t \rightarrow q\end{matrix}\right] \rightarrow q$$ and the proof is then complete.

This may also be proved by a truth table.

Example: Suppose that a collection of nickels and dimes has a value of 175 cents. Prove "There is an odd number of nickels."

Solution: Show that $\sim q$ "There is an even number of nickels: is false by reasoning until you encounter a statement which is inconsistent with known facts.

If the number of nickels is even, then the value of the nickels (expressed in cents) is a natural number ending in 0, then the value of the dimes (expressed in cents) must end in a natural number ending in 5. But the value of dimes will (in cents) end in 0. Therefore, $\sim q$ is false and q (number of nickels is even) is true.

Exercises:

1. Suppose that there are 13 people in a room. Prove: "At least two of these people were born in the same month. Use the indirect method.
2. Suppose that the number of trees in Illinois at a given time is greater than the number of leaves on any tree in Illinois, and that every tree has at least one leaf. Prove by the indirect method that there must be at least two trees with the same number of leaves.

Negation

Exercises

3. Suppose that a pen contains only cows and chicken, each of which is normal in every way. Prove the statement "There is an even number of feet in the pen."